

REEB ORBITS TRAPPED BY DENJOY MINIMAL SETS

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ABSTRACT. Let φ be any flow on T^n obtained as the suspension of a diffeomorphism of T^{n-1} and let \mathcal{A} be any compact invariant set of φ . We realize $(\mathcal{A}, \varphi|_{\mathcal{A}})$ up to reparametrization as an invariant set of the Reeb flow of a contact form on \mathbb{R}^{2n+1} equal to the standard contact form outside a compact set and defining the standard contact structure on all of \mathbb{R}^{2n+1} . This generalizes the construction of Geiges, Röttgen and Zehmisch.

1. INTRODUCTION

In [1] Eliashberg and Hofer proved that if α is a contact form on \mathbb{R}^3 which coincides with the standard contact form outside a compact set and defines the standard contact structure on all of \mathbb{R}^3 and if the Reeb flow of α has a bounded forward orbit, then the flow necessarily has a periodic orbit. Recently, Geiges, Röttgen and Zehmisch [3] (see also Röttgen [5]) showed that the higher dimensional analogue of the Eliashberg-Hofer theorem is not true. They constructed, for any $n \geq 2$, a contact form on \mathbb{R}^{2n+1} equal to the standard contact form outside a compact set and defining the standard contact structure on whole \mathbb{R}^{2n+1} , with the property that its Reeb flow has a bounded forward orbit but has no periodic orbits. In their example bounded forward orbits are trapped by an invariant n -dimensional torus with an ergodic linear flow.

In this note we generalize their construction. Our result enables us to produce diverse compact invariant sets in Reeb flows. Let $(\theta_1, \dots, \theta_n)$ be the standard angle coordinates of the n -dimensional torus T^n . We denote the coordinates of \mathbb{R}^{2n} (resp. \mathbb{R}^{2n+1}) by $(x_1, y_1, \dots, x_n, y_n)$ (resp. $(x_1, y_1, \dots, x_n, y_n, z)$), and the polar coordinates in the (x_j, y_j) -plane by (r_j, θ_j) . Via a natural inclusion using polar coordinates, we identify T^n with the subset $\mathbb{T} = \{r_1 = \dots = r_n = 1\}$ of \mathbb{R}^{2n} .

We show the following

Theorem 1.1. *Let V be any vector field on \mathbb{T} satisfying $\sum_{j=1}^n d\theta_j(V) > 0$, and let $\mathcal{A} \subset \mathbb{T}$ be any compact invariant set of the flow generated by V . Then, one can find a C^∞ contact form α on \mathbb{R}^{2n+1} which defines the standard contact structure ξ_{st} and satisfies the following properties:*

- (1) α equals the standard contact form α_{st} outside a compact neighborhood of $\mathcal{A} \times \{0\}$.
- (2) The flow φ generated by the Reeb vector field R of α has $\mathcal{A} \times \{0\}$ as its invariant set, and there exists a positive C^∞ function f on \mathbb{T} such that $R = fV$ on $\mathcal{A} \times \{0\}$.
- (3) All orbits of φ in the complement of $\mathcal{A} \times \{0\}$ in \mathbb{R}^{2n+1} are unbounded.

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- (4) *There exists an orbit of φ which is bounded in forward time and unbounded in backward time.*

Remark. On \mathbb{T} , the plane field $\xi_{\text{st}} \cap T(\mathbb{T})$ is expressed by $\sum_{j=1}^n d\theta_j = 0$ (See §2 for the definition of ξ_{st}). Thus, V in the above theorem covers all vector fields on \mathbb{T} positively transverse to ξ_{st} .

Let ψ be a flow on a manifold M . We say in this note that a compact minimal set \mathcal{M} of ψ is a *Denjoy minimal set* if for any point p of \mathcal{M} there exists a codimension 1 disk D in M transverse to ψ passing through p such that $D \cap \mathcal{M}$ is homeomorphic to a Cantor set. Combining Hall's result [4] with our theorem, we obtain the following

Corollary 1.2. *For any $n \geq 3$, there exists a C^∞ contact form α on \mathbb{R}^{2n+1} which defines the standard contact structure ξ_{st} and satisfies the following properties:*

- (1) *α equals α_{st} outside a compact set.*
- (2) *The flow φ generated by the Reeb vector field of α admits a Denjoy minimal set \mathcal{M} . And \mathcal{M} is the unique compact invariant set for φ (in particular, there are no periodic orbits).*
- (3) *There exists an orbit of φ which is bounded in forward time and unbounded in backward time (in this case the ω -limit set of this orbit is \mathcal{M}).*

2. BASIC DEFINITIONS AND FACTS

A C^∞ 1-form α on a $(2n+1)$ -dimensional manifold M is a *contact form* if $\alpha \wedge (d\alpha)^n$ never vanishes on M . A codimension 1 plane field ξ on M is a *contact structure* on M if $\xi = \ker \alpha$ for some contact form α . A manifold with a contact structure is called a *contact manifold*. Given a contact form α the *Reeb vector field* of α is the unique vector field R satisfying $\alpha(R) = 1$ and $i(R)d\alpha = 0$. For a contact manifold (M, ξ) a vector field X on M is a *contact vector field* of ξ if the flow generated by X preserves ξ . We denote by $C^\infty(M)$ the set of all real valued C^∞ functions on M , by $\Gamma^\infty(\xi)$ the set of all C^∞ vector fields on M tangent to ξ at each point of M , and by $\Gamma_\xi^\infty(TM)$ the set of all C^∞ contact vector fields on M .

The following two facts are fundamental (see [2]).

Proposition 2.1. *Let ξ be a C^∞ contact structure on M and α a C^∞ contact form such that $\xi = \ker \alpha$. Then, there is a bijective correspondence between $\Gamma_\xi^\infty(TM)$ and $C^\infty(M)$ given as follows: Define $\Phi : \Gamma_\xi^\infty(TM) \rightarrow C^\infty(M)$ by $\Phi(X) = \alpha(X)$ and $\Psi : C^\infty(M) \rightarrow \Gamma_\xi^\infty(TM)$ by $\Psi(H) = HR + Y$, where R is the Reeb vector field of α and Y is the unique vector field in $\Gamma^\infty(\xi)$ such that $i(Y)d\alpha = dH(R)\alpha - dH$. Then Φ and Ψ are the inverses of each other.*

Proposition 2.2. *Let α be a C^∞ contact form on M and let $\xi = \ker \alpha$. If $X \in \Gamma_\xi^\infty(TM)$ and if X is transverse to ξ everywhere on M , then X is the Reeb vector field of the contact form $\alpha/\alpha(X)$.*

We call

$$(2.1) \quad \alpha_{\text{st}} = dz + \frac{1}{2} \sum_{j=1}^n (x_j dy_j - y_j dx_j) = dz + \frac{1}{2} \sum_{j=1}^n r_j^2 d\theta_j$$

the *standard contact form* on \mathbb{R}^{2n+1} and $\xi_{\text{st}} = \ker \alpha_{\text{st}}$ the *standard contact structure* on \mathbb{R}^{2n+1} . The co-orientation of ξ_{st} is given by α_{st} . It is easy to see that the Reeb

vector field of α_{st} is ∂_z . The vector field Y in Proposition 2.1 with respect to α_{st} is expressed as follows (at points with $r_j \neq 0$).

$$(2.2) \quad Y = \sum \left[\left(\frac{r_j}{2} H_z - \frac{H_{\theta_j}}{r_j} \right) \partial_{r_j} + \frac{H_{r_j}}{r_j} \left(\partial_{\theta_j} - \frac{r_j^2}{2} \partial_z \right) \right]$$

Here, $\left\{ \partial_{r_j}, \partial_{\theta_j} - \frac{r_j^2}{2} \partial_z \right\}_j$ is a frame of ξ_{st} .

3. CONDITIONS ON X AND H

From now on, we exclusively consider the standard contact structure ξ_{st} on \mathbb{R}^{2n+1} . Take C^∞ functions k_j ($j = 1, \dots, n$) on T^n satisfying the condition

$$(3.1) \quad \sum_{j=1}^n k_j = 1.$$

We denote by U the subset of \mathbb{R}^{2n} consisting of points with $r_1 r_2 \cdots r_n \neq 0$. Then, by composing k_j with the natural projection $U \rightarrow T^n$ in polar coordinates, we may regard k_j as functions on U (although in general we cannot extend the domain to whole \mathbb{R}^{2n}). Let $\mathcal{A} \subset T^n$ be a compact invariant set for the flow generated by the vector field $\sum_{j=1}^n k_j \partial_{\theta_j}$. Let $\lambda > 0$ be a constant.

We consider the following conditions for a C^∞ contact vector field X of ξ_{st} .

- (X1) X is positively transverse to ξ_{st} at each point of \mathbb{R}^{2n+1} .
- (X2) $X = \partial_z$ outside a compact neighborhood of $\mathcal{A} \times \{0\}$.
- (X3) $dz(X) > 0$ outside $\mathcal{A} \times \{0\}$.
- (X4) $X = 2\lambda \sum_{j=1}^n k_j \partial_{\theta_j}$ on $\mathcal{A} \times \{0\}$.

Next, we consider the following conditions for a C^∞ function H on \mathbb{R}^{2n+1} .

- (H1) H is positive at each point of \mathbb{R}^{2n+1} .
- (H2) $H = 1$ outside a compact neighborhood of $\mathcal{A} \times \{0\}$.
- (H3) $H - \frac{1}{2} \sum_j r_j H_{r_j} > 0$ outside $\mathcal{A} \times \{0\}$.
- (H4) On $\mathcal{A} \times \{0\}$, we have $H = \lambda$, $H_{r_j} = 2\lambda k_j$, $H_{\theta_j} = 0$ ($j = 1, \dots, n$), and $H_z = 0$.

Then, via the correspondence (Proposition 2.1) these two sets of conditions are closely connected with each other:

Lemma 3.1. *Suppose a C^∞ function H on \mathbb{R}^{2n+1} satisfies the conditions (H1) to (H4). Then, the contact vector field X of ξ_{st} which corresponds with H via α_{st} satisfies the conditions (X1) to (X4).*

Proof. By Proposition 2.1, (H1) implies (X1). Suppose H satisfies (H2) with a compact neighborhood W . Then, since all the partial derivatives of H vanish outside W , by (2.2) we have $Y = 0$ hence $X = H\partial_z + Y = \partial_z$ outside W . This shows (X2). By (2.2) we have $dz(X) = dz(H\partial_z + Y) = H + dz(Y) = H - \frac{1}{2} \sum_j r_j H_{r_j}$. Thus, (X3) implies (X3). Finally, suppose H satisfies (H4). Then (X4) follows from (2.2) and (3.1) by a simple computation. \square

4. CONSTRUCTING H

This section is devoted to the construction of a function H on \mathbb{R}^{2n+1} satisfying the conditions (H1) to (H4). Let k_j , \mathcal{A} and U be as in the preceding section. For a real number b and τ we consider the following function $Q^b[\tau]$ on U .

$$(4.1) \quad Q^b[\tau](r_1, \dots, r_n, \theta_1, \dots, \theta_n) = \sum_i k_i(\theta_1, \dots, \theta_n)(r_i - \tau)^2 + b \sum_{p \neq q} (r_p - r_q)^2.$$

Let \mathbf{r} be the column vector with entries r_1, \dots, r_n and $A(b)$ the $n \times n$ matrix with (i, i) -th entry $k_i + 2(n-1)b$ and (i, j) -th entry $-2b$ for $i \neq j$. Then $Q^b[0] = {}^t \mathbf{r} A(b) \mathbf{r}$.

Lemma 4.1. *If b is sufficiently large, then $Q^b[\tau]$ is non-negative everywhere on U .*

Proof. It suffices to show that the matrix $A(b)$ is positive definite. If b is sufficiently large, the $s \times s$ principal minor ($s \leq n-1$) of $A(b)$ is positive because it is a polynomial in b of degree s with the coefficient of the highest order term being a positive constant. Also, $\det A(b)$ is positive because by (3.1) it is a polynomial in b of degree $n-1$ with the coefficient of the highest order term being a positive constant. \square

From now on, we always take b large so that the conclusion of Lemma 4.1 holds. Here, we briefly observe some properties of $A(b)$ from linear algebra. Let us denote the eigenvalues of $A(b)$ by $\lambda_1(b), \dots, \lambda_n(b)$. These are functions on T^n . Since $\frac{k_i}{2b}$ tends to 0 uniformly on T^n as $b \rightarrow \infty$, the matrix $\frac{1}{2b} A(b)$ converges uniformly to the matrix with diagonal entries $n-1$ and the other entries -1 , whose eigenvalues are 0 and n . The eigenspace W_0 corresponding to 0 is a 1-dimensional subspace of \mathbb{R}^n generated by the vector $(1, \dots, 1)$ and the eigenspace W_n corresponding to n is the orthogonal complement of W_0 . Thus, renumbering the eigenvalues if necessary, we may assume that as $b \rightarrow \infty$ the ratios of the eigenvalues of $A(b)$ converge uniformly as follows: $\lambda_1(b)/\lambda_i(b) \rightarrow 0$ ($i > 1$), $\lambda_i(b)/\lambda_j(b) \rightarrow 1$ ($i, j > 1$), and that, in the Grassmanian space, the eigenspace corresponding to $\lambda_1(b)$ converges to W_0 and the sum of the eigenspaces corresponding to $\lambda_i(b)$ ($i \geq 2$) converges to W_n .

For any $c > 0$, let $E(c)$ be the subset $\{\mathbf{v} \in \mathbb{R}^n \mid {}^t \mathbf{v} A(b) \mathbf{v} \leq c\}$ of \mathbb{R}^n and $J(c)$ the closed line segment joining two points $\pm(\sqrt{c}, \dots, \sqrt{c})$. We note that $E(c)$ depends on $(\theta_1, \dots, \theta_n) \in T^n$. Since we are assuming that $A(b)$ is positive definite, $E(c)$ is compact.

Lemma 4.2. *As b goes to infinity, $E(c)$ converges to $J(c)$ uniformly on T^n with respect to the Hausdorff distance.*

Proof. Let ℓ be the 1-dimensional linear subspace of \mathbb{R}^n generated by $(1, \dots, 1)$. Then, by (3.1) we have $E(c) \cap \ell = J(c)$ for any b . This with the observation made above implies that $E(c)$ converges uniformly to $J(c)$. \square

We write L for the subset of U consisting of all points satisfying the following condition: there exists i ($1 \leq i \leq n$) such that $\frac{1}{3} < r_i < \frac{2}{3}$ and that $\frac{1}{3} < r_j$ for all j ($j \neq i$).

Lemma 4.3. *Suppose $0 < C < 7/9$. If b is sufficiently large, then we have $Q^b[2] > 1 + C$ on L .*

Proof. Let P^b be the domain in U satisfying the condition $Q^b[2] \leq 1 + C$, and J the subset of U defined by $2 - \sqrt{1 + C} \leq r_1 = \dots = r_n \leq 2 + \sqrt{1 + C}$. Then, by Lemma 4.2, P^b converges to J as $b \rightarrow \infty$. Since $\frac{2}{3} < 2 - \sqrt{1 + C}$, the closure of L is disjoint from J , hence also from P^b for very large b . This shows the conclusion. \square

Here we prepare two more auxiliary functions. The first one is a monotone increasing C^∞ function $\rho : [0, \infty) \rightarrow [0, 1]$ such that

- ($\rho 1$) $\rho(r) = 0$ if and only if $r \leq 1/3$, and
- ($\rho 2$) $\rho(r) = 1$ if and only if $r \geq 2/3$.

The second one is a C^∞ function μ on T^n such that

- ($\mu 1$) $\mu = \mu_{\theta_j} = 0$ ($j = 1, \dots, n$) on \mathcal{A} , and
- ($\mu 2$) $\mu > 0$ on $T^n - \mathcal{A}$.

Such a μ exists.

Now, let C and λ be constants satisfying $0 < C < 7/9$ and $1 < \lambda < e^C$, and b a constant satisfying the conclusions of Lemmas 4.1 and 4.3. We define a C^∞ function K on \mathbb{R}^{2n+1} as follows:

$$(4.2) \quad K = \lambda \exp \left\{ \left[1 + C - Q^b[2] - (z^2 + \mu) \sum_j r_j \right] \prod_{\ell=1}^n \rho(r_\ell) - C \right\}.$$

Notice that, although the polar coordinate functions r_j and θ_j are defined only on $U \times \mathbb{R}$, by ($\rho 1$) K is well-defined on whole \mathbb{R}^{2n+1} .

We have

Lemma 4.4. K satisfies the conditions (H1), (H3) and (H4).

Proof. Obviously K is positive everywhere, thus satisfies (H1). On $\mathcal{A} \times \{0\}$, by (3.1), ($\mu 1$) and ($\rho 2$), we immediately obtain that $K = \lambda$, $K_{\theta_j} = 0$ ($j = 1, \dots, n$) and $K_z = 0$. Also, by ($\rho 2$), we have on $\mathcal{A} \times \{0\}$, $K_{r_j} = -K Q^b[2]_{r_j} = 2\lambda k_j$, showing that K satisfies (H4). Finally, let us check the condition (H3). Since $K - (1/2) \sum_j r_j K_{r_j} = (K/2)(2 - \sum_j r_j (\log K)_{r_j})$, it is enough to show that $\sum_j r_j (\log K)_{r_j} < 2$ outside $\mathcal{A} \times \{0\}$. We have

$$(4.3) \quad \begin{aligned} & \sum_j r_j (\log K)_{r_j} \\ &= \left[- \sum_j r_j Q^b[2]_{r_j} - (z^2 + \mu) \sum_j r_j \right] \prod_\ell \rho(r_\ell) \\ &+ \left[1 + C - Q^b[2] - (z^2 + \mu) \sum_j r_j \right] \sum_j r_j \rho'(r_j) \prod_{\ell \neq j} \rho(r_\ell). \end{aligned}$$

Here we note that

$$(4.4) \quad \sum_j r_j Q^b[2]_{r_j} = 2Q^b[1] - 2.$$

We consider three cases separately.

Case 1. $r_j \leq 1/3$ for some j . Then, since $\rho(r_j) = \rho'(r_j) = 0$, the RHS of (4.3) is 0.

Case 2. $r_j \geq 2/3$ for all j . Then, the RHS of (4.3) is $-\sum_j r_j Q^b[2]_{r_j} - (z^2 + \mu) \sum_j r_j$. This with $(\mu 2)$, (4.4) and the positive definiteness of Q , implies the desired conclusion.

Case 3. $1/3 < r_j < 2/3$ for some j . In this case, again by (4.4) and the positive definiteness of Q we see that the first term of the RHS of (4.3) is less than 2. On the other hand, Lemma 4.3 implies that the second term is non-positive. So the conclusion follows.

This finishes the proof that K satisfies (H3). \square

Lemma 4.5. K is not greater than λe^{-C} outside some compact region.

Proof. In the case where $r_j \leq 1/3$ for some j , we have $K = \lambda e^{-C}$. Let us consider the case where $r_j > 1/3$ for all j . We denote by T the quantity in the square bracket in (4.2) and claim that T is non-positive outside some compact region. In fact, since $Q^b[2]$ and μ are non-negative and $\sum_j r_j > n/3$, T is non-positive for $z^2 \geq \frac{3(1+C)}{n}$. T is also non-positive for $\|\mathbf{r}\|$ large because the proof of Lemma 4.3 implies that $\sup\{\|\mathbf{r}\| \mid Q^b[2](\mathbf{r}) \leq 1 + C\}$ is finite. This shows the claim, and the Lemma follows. \square

K constructed above does not satisfy (H2). By a modification we will improve K so as to satisfy all the required conditions. To this end we need one more auxiliary function. It is a monotone increasing C^∞ function $G : (0, \infty) \rightarrow (0, \infty)$ such that

- (G1) $G(t) = t$ near $t = \lambda$,
- (G2) $G(t) = 1$ for $t \leq \lambda e^{-C}$, and
- (G3) $t(\log G)'(t) \leq 1$ for all t .

Since $\lambda e^{-C} < 1$, one can easily give such a G . Now, set $H = G \circ K$. Then, we have the following, as desired.

Lemma 4.6. H satisfies the conditions (H1) to (H4).

Proof. Clearly H satisfies (H1). The validity of (H4) for H follows from that for K because by (G1) we have $H = K$ in a neighborhood of $\mathcal{A} \times \{0\}$. (H2) follows from Lemma 4.5 and (G2). As stated in the proof of Lemma 4.4, in order to show (H3) it suffices to prove that $\sum_j r_j (\log H)_{r_j} < 2$ outside $\mathcal{A} \times \{0\}$. We have

$$\sum_j r_j (\log H)_{r_j} = \sum_j r_j (\log G(K))_{r_j} = \sum_j r_j (\log K)_{r_j} K (\log G)'(K).$$

This is indeed less than 2, because we have already seen in the proof of Lemma 4.4 that $\sum_j r_j (\log K)_{r_j} < 2$ and by (G3) we have that $K (\log G)'(K) \leq 1$. This verifies (H3). \square

5. REALIZING INVARIANT SETS IN REEB FLOWS

Proof of Theorem 1.1. Suppose that V is a vector field on T^n satisfying $\sum_{j=1}^n d\theta_j(V) > 0$, and that $\mathcal{A} \subset T^n$ is a compact invariant set of the flow generated by V . Then, if we write $V = \sum_j \nu_j \partial_{\theta_j}$, the sum $\sum_\ell \nu_\ell$ of the coefficient functions is positive. Take any constants C and λ such that $0 < C < 7/9$ and $1 < \lambda < e^C$, and define C^∞ functions k_j and f on T^n by $k_j = \frac{\nu_j}{\sum_\ell \nu_\ell}$ and $f = \frac{2\lambda}{\sum_\ell \nu_\ell}$. Then, we

have $\sum_j k_j = 1$ and $fV = 2\lambda \sum_j k_j \partial_{\theta_j}$. By using these λ and k_j , we can construct H satisfying the conditions (H1) to (H4) just as in §4. If we write X for the contact vector field corresponding with H with respect to α_{st} and put $\alpha = \alpha_{\text{st}}/H$, we see from Lemma 3.1 that X is the Reeb vector field of α and satisfies (X1) to (X4). All the conclusions of Theorem 1.1 now follow immediately. \square

Proof of Corollary 1.2. In [4], Hall constructed a C^∞ embedding, say g , of a compact annulus A into itself admitting a Cantor set as minimal invariant set (Cantor minimal set, for short). If we choose an embedding $\iota : A \rightarrow T^2$ and extend $\iota \circ g \circ \iota^{-1} : \iota(A) \rightarrow \iota(A)$ appropriately, we obtain a C^∞ diffeomorphism h_2 of T^2 which is isotopic to the identity and admits a Cantor minimal set. For any $n \geq 3$, a diffeomorphism h_{n-1} of $T^{n-1} = T^2 \times T^{n-3}$ having a Cantor minimal set is defined by $h_{n-1} = h_2 \times \text{id}$. Now, we consider the suspension flow of h_{n-1} . It is a flow on T^n which has a Denjoy minimal set and whose orbits are transverse to the fibers of the natural projection $T^n \rightarrow T^{n-1}$. Passing to a conjugate flow ψ via a suitable linear automorphism of T^n we may assume that the vector field V associated to ψ are transverse to the plane field $\sum_{j=1}^n d\theta_j = 0$. Thus, we can apply Theorem 1.1 to V and obtain the conclusion. \square

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